A NOTE ON A CORRESPONDENCE PRINCIPLE IN NONLINEAR VISCOELASTIC MATERIALS

K. R. Rajagopal and A. R. Srinivasa
Department of Mechanical Engineering, Texas A&M University, College Station, TX 77843
e-mail:krajagopal@megr.tamu.edu

Abstract. We show that models for the nonlinear viscoelastic response of solids generated on the basis of a correspondence principle developed by Shapery (1984) do not satisfy the balance of angular momentum for large deformations. This principle, which is valid if the displacement gradients are sufficiently small, has been used in several papers to develop models to describe the fracture of viscoelastic solids, and these studies need to be reexamined in the light of this note.

Keywords: Angular momentum, correspondence principle, finite deformations, nonlinear viscoelasticity.

1. Introduction. The well known correspondence between the solutions of boundary value problems using linearized elasticity and those obtained using the theory of linearized viscoelastic materials (see e.g., Christensen (1982), Wineman and Rajagopal (2000)) has been gainfully exploited to solve a variety of problems in viscoelasticity. As the underlying theories are linear, superposition of solutions is possible and this has been one of the reasons for the success that this “correspondence principle” has enjoyed. In 1984, Shapery (1984) proposed “correspondence principles” to generate a class of models for nonlinear viscoelastic materials by first considering the constitutive equations for a nonlinear elastic material and using a convolution integral to develop a corresponding constitutive equation for a non-linear viscoelastic material. He has shown that if the stresses for the reference elastic material satisfies the balance of linear momentum, so does that of the corresponding viscoelastic material. Shapery (1984) then went on to study the fracture of nonlinear viscoelastic solids by appealing to his correspondence principle. This study of Shapery’s has been followed by many others that appeal to the correspondence principle to describe the response of nonlinear viscoelastic solids.

However, when large deformations are involved, the models that are generated using the “correspondence principle” will not, in general satisfy the balance of angular momentum. For example, one cannot use this principle if
large deformations involving shear, bending, torsion etc. which are commonly encountered in structures are involved.

Schapery(1984) does discuss another limitation of the model, namely that it does not satisfy frame-indifference and hence we shall not discuss this issue here. However, he has overlooked the fact that it does not satisfy the balance of angular momentum.

Although this note restricts attention to the correspondence principle CP1 of Schapery(1984), the results established also apply to the other correspondence principles that he has developed in this paper.

2. The correspondence principle. We shall restrict our attention to the most general of the “correspondence principles” namely that which is referred to as “CP1” by Schapery(1984). The starting point for Schapery(1984) in postulating his “correspondence principle” CP1 is the concept of a “reference elastic material” whose displacement is given by \( \mathbf{u}^R(x, t) \), where \( x \) represents the reference coordinates of the body and whose referential or non-symmetric Piola-Kirchhoff stress is given by \( \mathbf{\sigma}^R \). This reference material is assumed to obey the constitutive equations of finite elasticity with an appropriate frame invariant nonlinear strain being defined, so that there is a strain energy function \( W(u^R_{i,j}) \) (where the derivatives are with respect to reference coordinates) such that

\[
\sigma^R_{ij} = \frac{\partial W}{\partial u^R_{i,j}}.
\]

The relationship between the displacements and stresses of the actual viscoelastic material and the reference elastic material are given by (Schapery(1984), eqns (24) and (28)) the following convolutions

\[
u_i(x, t) = E_R \int_{-\infty}^{t} D(t-T, \tau) \frac{\partial}{\partial \tau} u^R_i(x, \tau) d\tau,
\]

and

\[
\sigma_{ij} = \frac{1}{E_{1R}} \int_{-\infty}^{t} \frac{E_1(t-\tau, t)}{E_1(t-\tau, t)} \frac{\partial}{\partial \tau} \sigma^R_{ij} d\tau = \int_{-\infty}^{t} \frac{E_1(t-\tau, t)}{E_1(t-\tau, t)} \frac{\partial}{\partial \tau} \frac{\partial W}{\partial u^R_{i,j}} d\tau,
\]

where \( E_R \) and \( E_{1R} \) are constants and \( D(t-\tau, t) \) and \( E_1(t-\tau, t) \) are appropriate kernels called the creep compliance and viscoelastic modulus respectively. It is a simple matter to show that (Schapery(1984)) if \( \sigma^R_{ij} \) satisfies the balance of linear momentum in the absence of body forces then so will \( \sigma_{ij} \), i.e., that

\[
\sigma^R_{ij,j} = 0 \Rightarrow \sigma_{ij,j} = 0.
\]
3. The balance of angular momentum

In this section, we state the conditions for the satisfaction of the balance of angular momentum and prove that the viscoelastic models generated by using Schapery's "correspondence principle" do not satisfy the balance of angular momentum.

Recall that the satisfaction of the balance of angular angular momentum in the absence of body couples demands that, in terms of the Piola-Kirchhoff stresses for the reference elastic material (see e.g., Ogden(1))

$$
\sigma_{ij}^R(\delta_k + u_k^{i,j}) = (\delta_{ij} + u_i^{R,j})\sigma_{k}^R,
$$

where $\delta_{ij}$ is the Kronecker delta. In direct notation,

$$
\sigma^R(F^R)^T = F^R(\sigma^R)^T,
$$

where the tensor $F^R$ is the deformation gradient of the reference material and is given in terms of the displacement gradient as $F^R_{ij} := \delta_{ij} + u_i^{R,j}$. Of course, this condition is easy to impose on the reference elastic material by simply choosing the strain energy function to be $W = W((F^R)^TF^R)$. In fact, for the case of elasticity, the frame-indifference of the strain energy function guarantees that the balance of angular momentum is met and that the Cauchy or true stress is symmetric. Also, Constitutive models might satisfy balance of angular momentum but not frame indifference. For example, both linearized elasticity and linearized viscoelasticity satisfy the balance of angular momentum when expressed in terms of the Cauchy stress although they do not satisfy frame indifference. Schapery(1984) recognized that while the reference model is frame indifferent, the viscoelastic model generated by (3) is not.

Let us assume that the strain energy function of the reference elastic material satisfies the requirement of frame-indifference and examine the satisfaction of the angular momentum balance for the viscoelastic material. For this latter material, the balance of angular momentum becomes

$$
\sigma_{ij}(\delta_k + u_{k,j}) = (\delta_{ij} + u_{i,j})\sigma_{k}^j,
$$

i.e., the necessary and sufficient condition for the balance of angular momentum to be met is that the tensor $\sigma F^T$ must be symmetric.

The source of the problem can be identified by noting that since both stresses and displacements for the viscoelastic materials are obtained by using convolution integrals with different kernels the condition (7) cannot be met in general. To see this, we use (2) and (3) and note that the left hand side of (7) becomes

$$
\left\{ \frac{1}{E_1R} \int_{-\infty}^{t} E_1(t-\tau,t) \frac{\partial}{\partial \tau} \sigma_{ij}^R d\tau \right\} \left\{ \delta_{kj} + E_R \int_{-\infty}^{t} D(t-s,t) \frac{\partial}{\partial s} u_k^{i,j}(x,s) ds \right\}.
$$

(8)
Clearly the above tensor is non-symmetric for general deformation histories since the terms involving $\sigma^R_{ij}$ and $u^R_{ij}$ cannot be grouped together. Moreover, the kernels in the two integrals are different (one is a relaxation modulus and the other is a creep compliance) so that there is no possibility for the result to be symmetric. It is for this reason that the viscoelastic model introduced through (2) and (3) violates the balance of angular momentum.

4. A simple counterexample. We now construct a simple counterexample utilizing a general isotropic elastic material as the elastic reference material and by considering a homogenous simple shear as the deformation of the reference material.

For an isotropic elastic material the Piola-Kirchhoff Stress is given by

$$\sigma^R = \frac{1}{\det(F^R)} \left\{ \alpha_0 (F^R)^{-T} + \alpha_1 F^R + \alpha_2 F^R (F^R)^T F^R \right\}, \quad (9)$$

where $\alpha_i$ are functions of the invariants of the strain and are obtained by differentiating the strain energy function with respect to the various invariants\(^1\).

For example, for a Hadamard compressible elastic material, the strain energy function is of the form

$$W = A(I_1 - 3) + B(I_2 - 3) + f(J), \quad (10)$$

where $I_1 = \text{tr}((F^R)^T F^R)$, $J = \det(F^R)$, and $I_2 = J^2 \text{tr}((F^R)^{-1}(F^R)^{-T})$. In these definitions the symbol $\text{tr}(\cdot)$ stands for the trace (i.e., the sum of the diagonals), and where $A$ and $B$ are constants, and $f(J)$ is a suitably chosen function of $J$. For such a material,

$$\alpha_0 = 2Jf'(J), \quad \alpha_1 = \frac{2}{J}(A + BI_1), \quad \alpha_2 = -\frac{2}{J}B. \quad (11)$$

Now let the isotropic reference elastic material be subject to a homogenous simple shear deformation, with the displacement given by

$$u^R = k(t)x_2 \mathbf{i}, \quad (12)$$

where $k(t)$ is the amount of shear and $\mathbf{i}$ is the unit vector along the $x-$axis. It is a simple matter to compute the deformation gradient matrix as

$$F^R = \begin{pmatrix}
1 & k(t) & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}. \quad (13)$$

\(^1\)The precise form of the coefficients can be found in Ogden() but are not needed for the present discussion.
In the present case, we note from (13) that \( \det(F^r) = 1 \) and that the only kinematical parameter that appears in the problem is \( k(t) \) so that, for any isotropic elastic material \( \alpha_i = \alpha_i(k(t)) \). Now substituting (13) into (9) we obtain

\[
\sigma^R = \alpha_0(k) \begin{pmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \alpha_1(k) \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \alpha_2(k) \begin{pmatrix} 1 + k^2 & k(2 + k^2) & 0 \\ k & 1 + k^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\] (14)

Since the deformation is homogeneous, it is obvious that equilibrium is satisfied (in the absence of body forces). It can be easily verified by postmultiplying (14) by the transpose of (13) that the resulting matrix is symmetric and hence the Piola-Kirchhoff stress of the reference material satisfies the balance of angular momentum, i.e., (6).

Now, let us compute the stress and deformation gradient for the actual viscoelastic material. Using (2) together with (12), we obtain

\[
\mathbf{u}(t) = g(t) x_2 \mathbf{i}, \quad g(t) := E_R \int_{-\infty}^{t} D(t - \tau, t) \frac{dk(\tau)}{d\tau} d\tau,
\] (15)

i.e., the actual motion of the viscoelastic body is also a homogeneous simple shear and the deformation gradient is computed to be

\[
\mathbf{F} = \begin{pmatrix} 1 & g(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\] (16)

Furthermore, the stress associated with the viscoelastic material can be computed from (14) and (3) as

\[
\sigma = \begin{pmatrix} a(t) & b(t) & 0 \\ c(t) & a(t) & 0 \\ 0 & 0 & d(t) \end{pmatrix},
\] (17)
where

\[
a(t) = \frac{1}{E_1R} \int_{-\infty}^{t} E_1(t - \tau, t) \frac{d}{d\tau} \{ \alpha_0(k(\tau)) + \alpha_1(k(\tau)) + \alpha_2(k(\tau))(1 + k(\tau)^2) \} d\tau,
\]

\[
b(t) = \frac{1}{E_1R} \int_{-\infty}^{t} E_1(t - \tau, t) \frac{d}{d\tau} \{ k(\tau)\alpha_1(k(\tau)) + k(\tau)\alpha_2(k(\tau))(2 + k(\tau)^2) \} d\tau,
\]

\[
c(t) = \frac{1}{E_1R} \int_{-\infty}^{t} E_1(t - \tau, t) \frac{d}{d\tau} \{ k(\tau)(\alpha_2(k(\tau)) - \alpha_0(k(\tau)) \} d\tau,
\]

\[
d(t) = \frac{1}{E_1R} \int_{-\infty}^{t} E_1(t - \tau, t) \frac{d}{d\tau} \{ \alpha_0(k(\tau)) + \alpha_1(k(\tau)) + \alpha_2(k(\tau)) \} d\tau.
\]

Now a routine calculation reveals that

\[
\sigma F^T \neq F \sigma^T
\]

unless \( a(t)g(t) + c(t) = b(t) \), which is not true in general.

For example, for the Hadamard material considered in (10), we note that for the deformation considered here, \( \alpha_0 \) and \( \alpha_2 \) are constants and \( \alpha_1 = 2(A + (k^2 + 3)B) \). Thus, substituting these into the equations (18-20) and simplifying we find that \( \alpha_0 \) is contained only in \( c(t) \), and thus for the Hadamard reference material, \( a(t)g(t) + c(t) \neq b(t) \). It is easy to construct many such counterexamples. In fact, given any material response of the form (3) one can always find a deformation history such that the corresponding stress response violates (8) and hence violates the balance of angular momentum.

References


